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Del Centina, Andrea (I-FERR-NDM); **Fiocca, Alessandra** (I-FERR-NDM)
Boscovich's geometrical principle of continuity, and the "mysteries of the infinity". (English summary)
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Roger Joseph Boscovich (1711–1787) is well known in the history of physics for his theory of atoms surrounded by fields of attracting and repelling forces. Invoking a principle of continuity, Boscovich reasoned that at a fundamental level mechanical interactions are governed by continuous forces acting at a distance between point atoms. He rejected the prevailing model of dynamical motion that posited particles interacting through inelastic collision, where continuous motion was theorized as a series of discrete infinitesimal impulses. Boscovich's *Theoria philosophiae naturalis* of 1758 was devoted to an exposition of his theory.

The article under review is a detailed study of two more purely mathematical works by Boscovich, *Sectionum conicarum elementa* and *De transformatione locorum geometricorum ubi de continuitatis lege, ac de quibusdam infiniti mysteriis* of 1754. As in his physical investigations, the concept of continuity occupied a fundamental place in his thinking. Geometrical figures were analyzed using a principle of continuity, according to which a given property of a figure will continue to hold as the figure undergoes a variation through a succession of states. Boscovich asserted, "It is wonderful that in geometry the continuity law is ever preserved, according to which nothing changes by leaps; no thing arises or vanishes at once, but always passes from one magnitude to another through all the intermediate stages" (p. 146).

A principle of continuity appeared in Johannes Kepler's writings on conic sections, in a letter written by Gottfried Leibniz and published in 1687, in Boscovich's works, and in Jean-Victor Poncelet's treatise of 1822 on projective geometry. Books on geometry that contained some ideas similar to those of Boscovich were published by Lazare Carnot in 1801 and 1803.

Boscovich's investigations were detailed and systematic. There is a wealth of interesting mathematical results and ideas in his geometrical writings, and the authors provide the most complete historical account to date of the subject. The article is a contribution to the history of geometry, to the intellectual biography of Boscovich, and to the history of mathematical ideas in the eighteenth century.

The following example—described by the authors on pp. 148–149—provides some indication of Boscovich's approach. If two line segments together make up the diameter of a circle, their mean proportion is obtained by erecting the perpendicular to the circle at the dividing point. (If c is the mean proportion of a and b then $c^2 = ab$.) The result was a well-known and fairly elementary proposition of plane geometry. It was presented by Boscovich as an equality between the rectangle formed by the line segments and the square on the perpendicular. Boscovich observed that if the dividing point is taken on the extension of the diameter outside the circle, one obtains a rectangle of negative area, and the side of the corresponding square will be imaginary (or "impossible" in the terminology of the period). There is evidently in this case no longer any circle or perpendicular. However, Boscovich was able to produce a simple geometrical construction using Euclid's *Elements III*, 36 that expressed the positive value of the area in terms of the square on the ordinate of a hyperbola. The transition from the construction involving the circle to the one involving the hyperbola involved the same functional relation between the line segments and the perpendicular, and it was in the permanence of this relation that the principle of continuity was manifested. The result in question was further developed and put into a more general form by Poncelet in his 1822 treatise. *Craig G. Fraser*

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